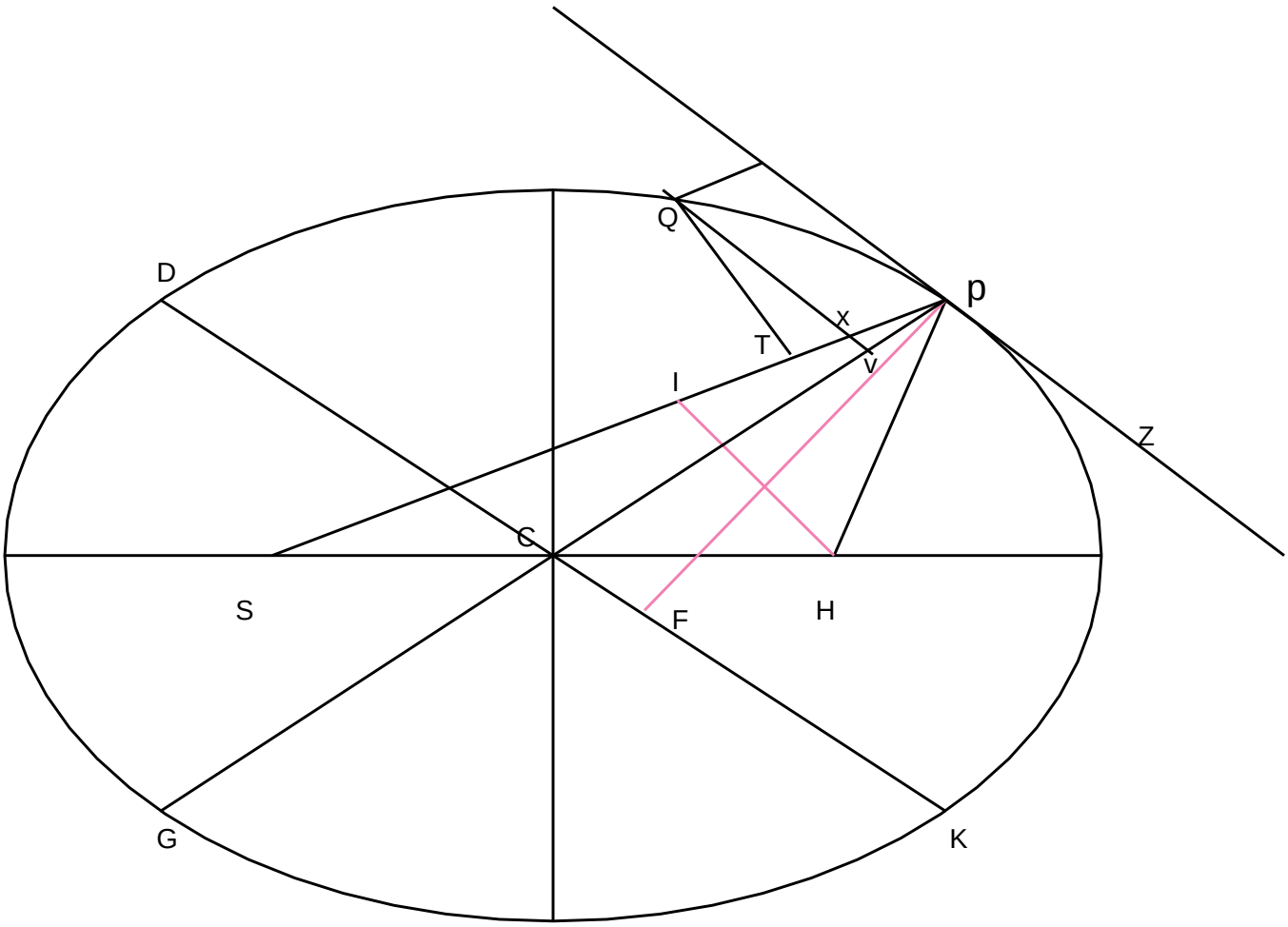
Section III: The Motion of bodies along conic section

Proposition XI Problem VI

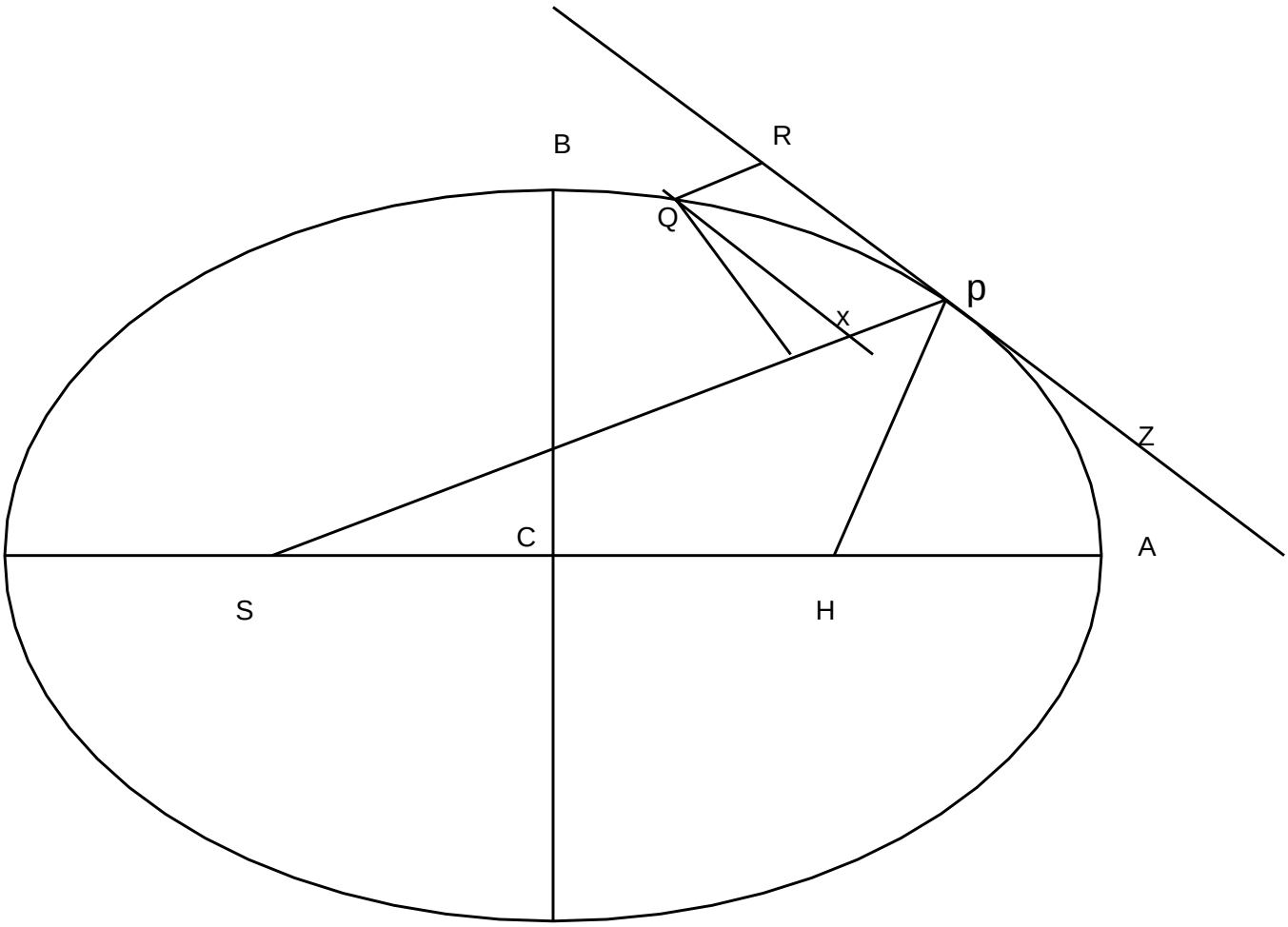
If a body revolves in an ellipse; it is required to find
the law of the centripetal force tending to the focus of the ellipse.



INTRODUCTION

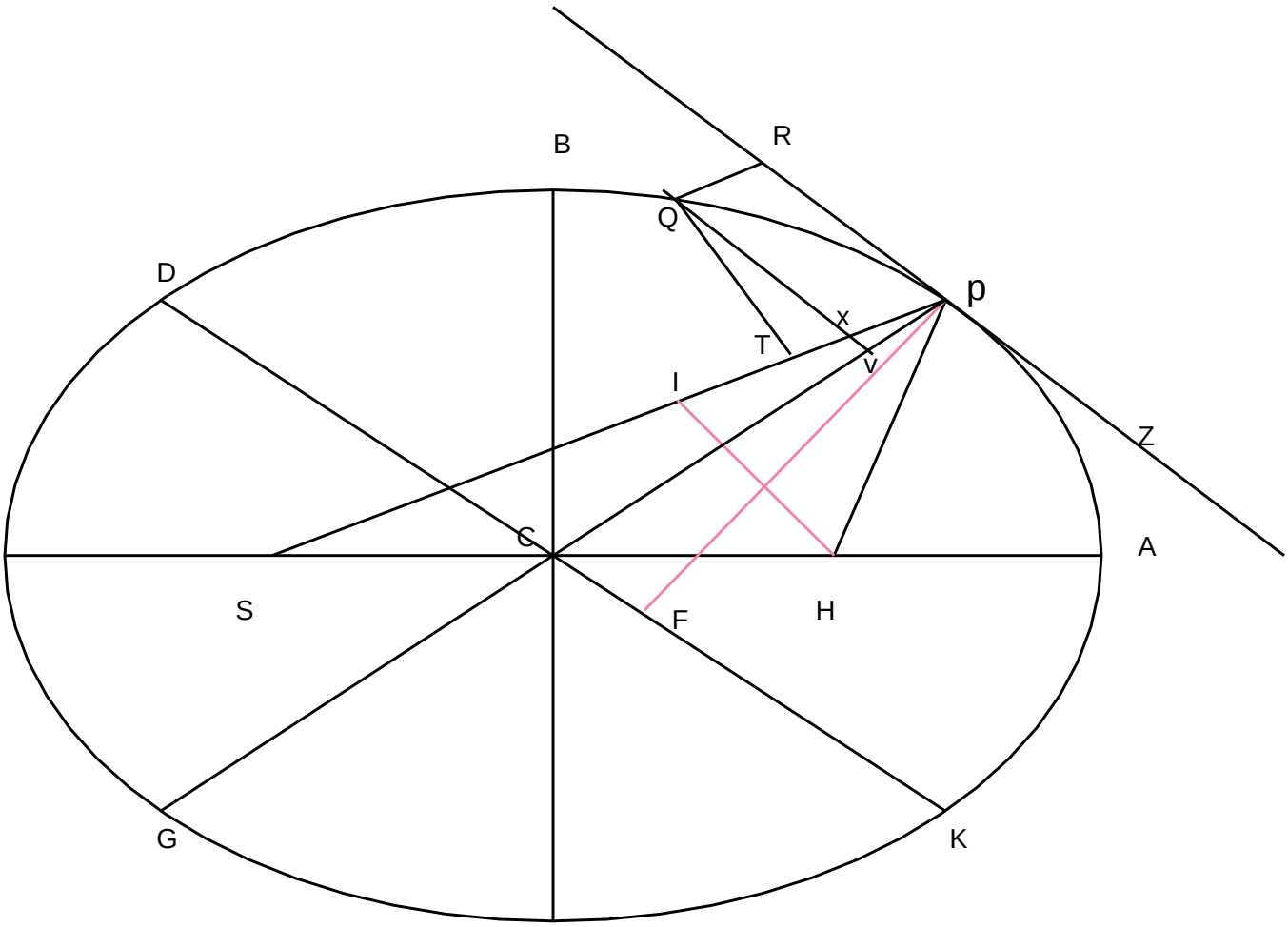
The propositions of Section III are commonly regarded as the apex of the Principia. In Proposition XI, the inverse-square law of attraction is deduced for bodies revolving in ellipses under the action of a 'a centripetal force tending to the focus of the ellipse.

Description of Figure



* S and H are the foci and C is the centre of the ellipse;
* CA and CB are the semimajor and
  semiminor axes of lengths a and b;
* P and Q are neighbouring points.
  Q --> P or Q goes to, becomes P eventually;
* RPZ is the tangent at P;

* DCK is the diameter, join together PCG
  and parallel to RPZ
* PF is perpendicular to DCK,
  and parallel to SP

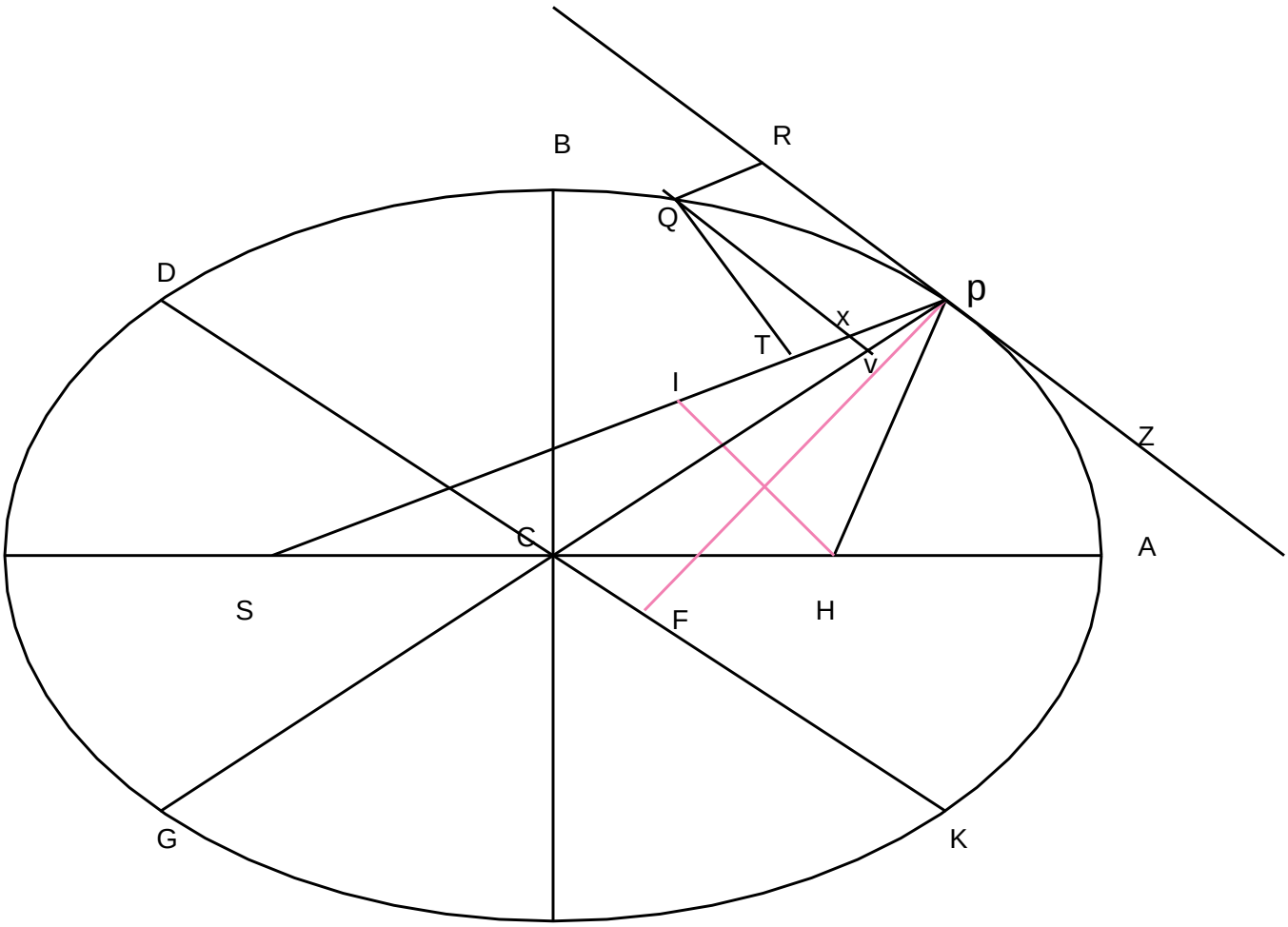


* QR and QT are pararell and perpendicular,
  respectively, to SP;
* Qv and IH are parallel to RPZ (and DCK)
* v is on PC and x is on PS

Properties of the Ellipse needed in the Solution

* SP + PH = 2a

* PF^2 * CD^2 = CB^2 * CA^2 (proposition X, equation (v))



Pv * $\frac{vG^2}{(Qv)^2} = \frac{CP^2}{CD^2}$ (Proposition X, equation (i))

* The latus rectum L = $\frac{2B^2}{a} = \frac{2BC^2}{CA}$ =
twice the semilatus rectum, m .

* ang IPR = ang HPZ implying ang PIH = ang PHI
and PI = PH

Thus.....

Since SC = CH and EC is parallel to IH, then

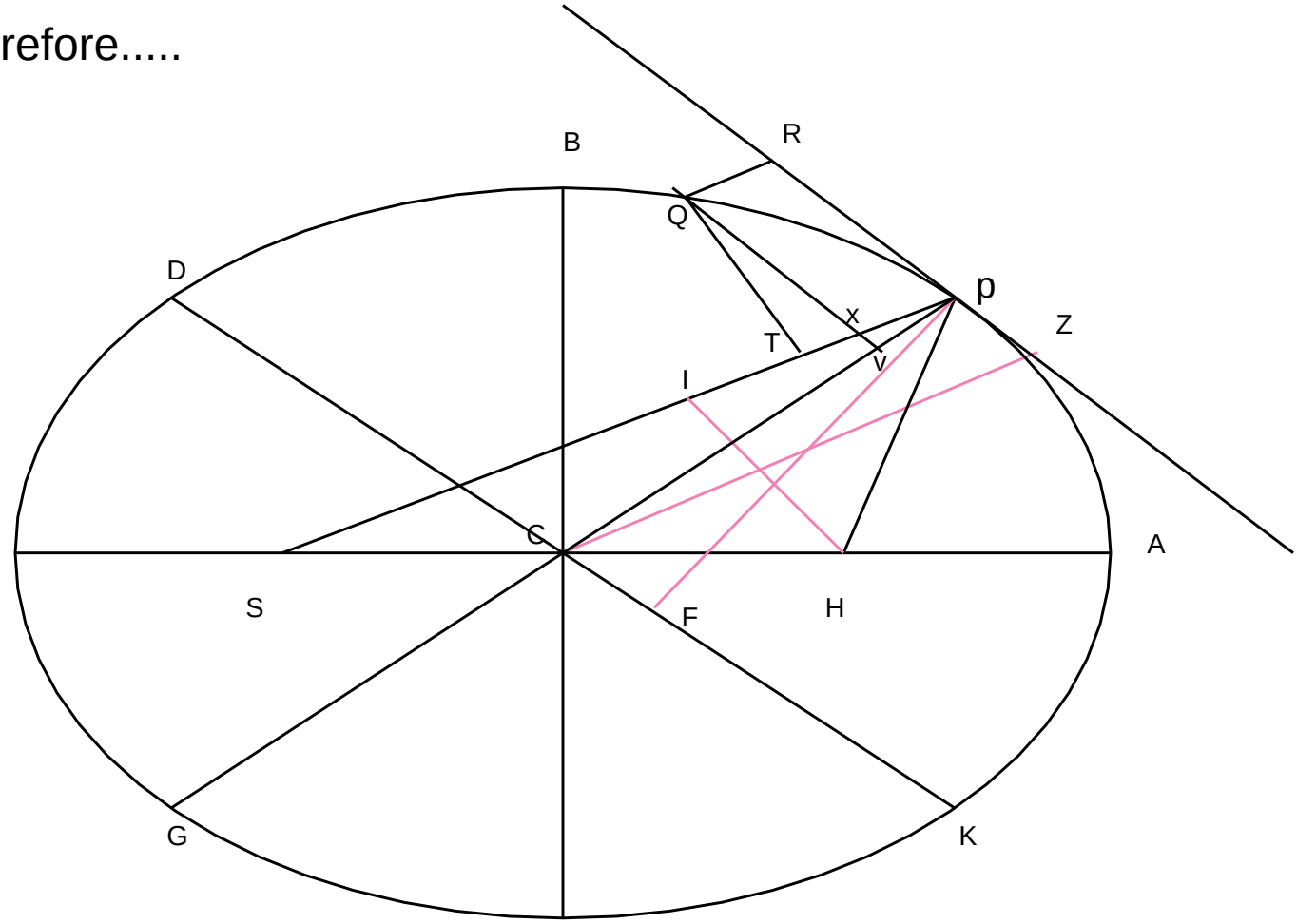
SE = EI

By previous stated relations that SP+PH = 2a,

PI = PH, and SE = EI

we have: 2a = PS + PH = PE + SE + PI = PE + EI + PI = 2PE

therefore.....



PE = CZ = a = AC

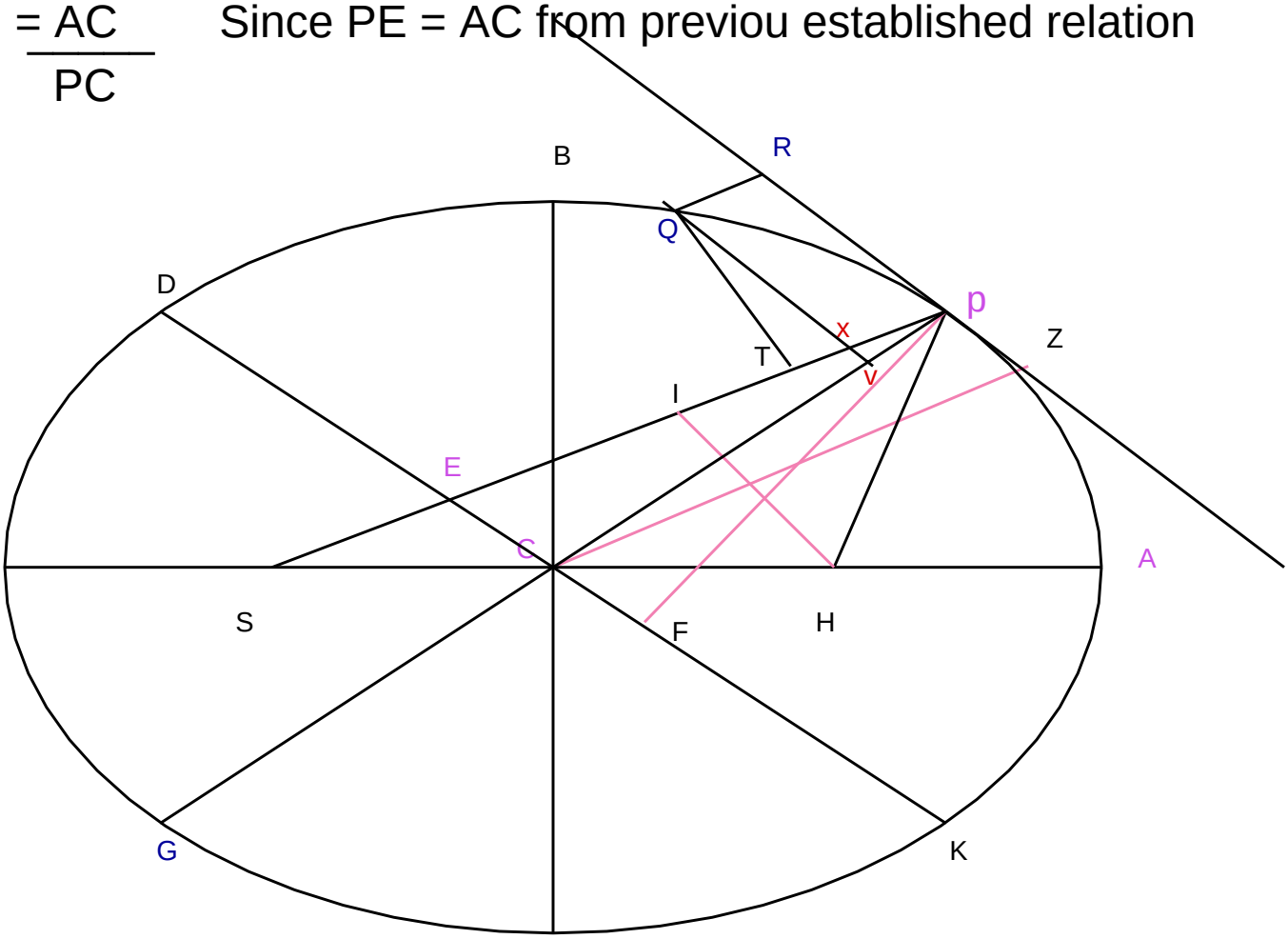
With these properties in mind, Newton's solution in finding the the law of centripetal force tending to the focus of the ellipse proceed as follow :

QRPx is a parallelogram thus,

$$\frac{QR}{Pv} = \frac{Px}{Pv}$$

$$= \frac{PE}{PC}$$ By the similar triangle of Pxv and PEC

$$= \frac{AC}{PC}$$ Since PE = AC from previou established relation



By a relation stated earlier, Prop. X : $\frac{Pv}{(Qv)^2} = \frac{CP^2}{CD^2 * vG}$

multiply with equation on the top : QR/Pv = AC/PC gives

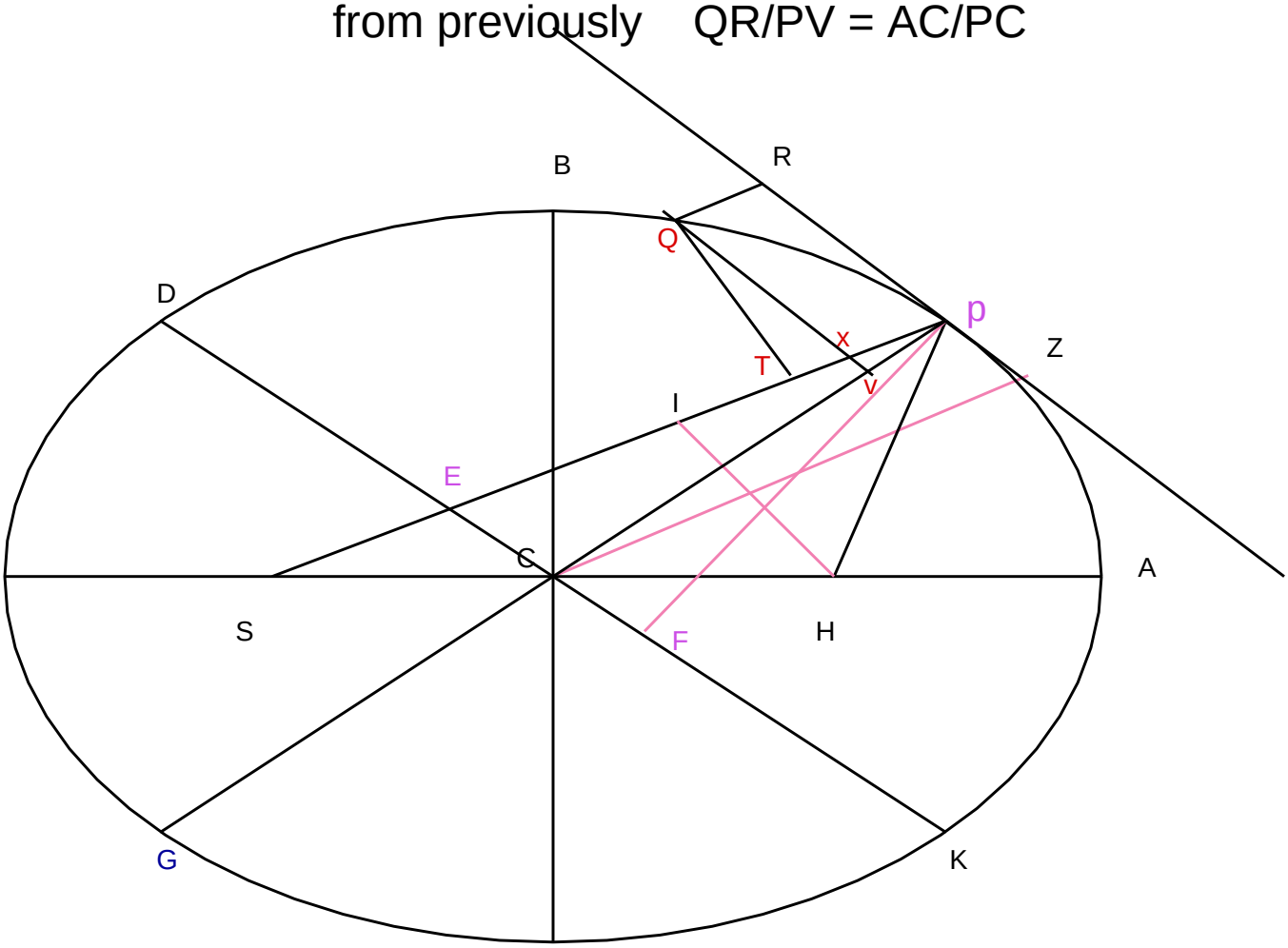
$$\frac{QR}{Qv^2} = \frac{AC*PC}{vG*CD^2}$$

Now,

$$\frac{(Qv)^2}{QT^2} \sim= \frac{(Qx)^2}{(QT)^2} = \frac{PE^2}{PF^2}$$ by similar triangle QxT and PEF

$$= \frac{AC^2}{PF^2}$$

from previously QR/PV = AC/PC



Since PF^2 * CD^2 = CB^2 * CA^2 as stated before, thus :

$$\frac{CD^2}{CB^2} = \frac{CA^2}{PF^2} \Rightarrow \frac{(Qv)^2}{QT^2} \sim= \frac{CA^2}{PF^2} \Rightarrow$$

$$\frac{(Qv)^2}{QT^2} \sim= \frac{CD^2}{CB^2}$$

multiply both side of

$$\frac{QR}{QT^2} \sim= \frac{AC*PC}{vG*CB^2}$$

by the latus rectum L, hence,

$$L * \frac{QR}{QT^2} \sim= \frac{2BC^2}{CA} * \frac{AC*PC}{vG*CB^2}$$

After cancelling

$$L * \frac{QR}{QT^2} \sim= 2\frac{PC}{vG} \sim= 1$$

Hence,

$$L * QR \sim= QT^2$$

and finally by Prop. VI, Corollary I showed in another page:

$$L * SP^2 \sim= \frac{SP^2 * QT^2}{QR} \sim= \frac{1}{C.F}$$

C.F = centripetal force

take the inverse on both side we get:

$$C.F \sim= SP^{-2}$$

that is, an inverse-square law of attraction (Q.E.I)